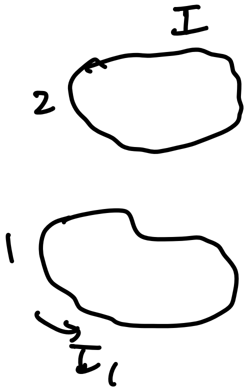
Lecture 26. Alternating current circuits

Last time:

* Mutual inductance M

$M \equiv$ proportionality constant between the mag flux induced on a loop due to a nearby loop



$\Phi_2 = M I_1$

$\Phi_1 = M I_2$

* Self inductance L

$\Phi = L I$ intrinsically geometrical positive.

* Magnetic energy stored in an inductor

$$\boxed{U_B = \frac{1}{2} L I^2}$$

* Kirchhoff's loop rule for inductors:

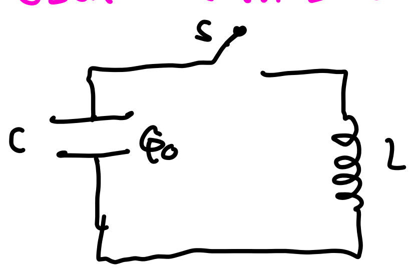
the "potential change" due to an inductor

$-L \left(\frac{dI}{dt}\right)$ if L is traversed in the direction of I

$L \left(\frac{dI}{dt}\right)$ if L is traversed in the direction opposite to I

Today: LC circuits and circuits with an AC source

Oscillations in LC circuit



circuit where the capacitor has charge Q_0 at $t<0$

At $t=0$ we close the switch $\rightarrow$ capacitor begins to discharge $\rightarrow$ current

$\rightarrow$ magnetic energy stored into inductor.

Since we don't have a resistance energy is transformed back & forth between capacitor & the inductor $\rightarrow$ electromagnetic oscillation.

The total energy in the LC circuit at $t>0$:

$$U = U_c + U_L = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2 \leftarrow$$

$$\frac{dU}{dt} = 0 \quad \text{no energy dissipated}$$

$$\Rightarrow \frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0$$

$$\Rightarrow \frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0 \quad \text{since } I = -\frac{dQ}{dt} \text{ and } \frac{dI}{dt} = -\frac{d^2Q}{dt^2}$$

We can also obtain this equation by applying Kirchhoff's rule:

$$\frac{Q}{C} - L\frac{dI}{dt} = 0$$

The solution for this eq is:

$$Q(t) = Q_0 \cos(\omega_0 t + \phi),$$

where
$Q_0 \equiv$ charge amplitude
$\phi \equiv$ phase
$\omega_0 \equiv$ angular frequency $= \frac{1}{\sqrt{LC}}$

We can now calculate the current:

$$I(t) = -\frac{dQ}{dt} = \omega_0 Q_0 \sin(\omega_0 t + \phi) = I_0 \sin(\omega_0 t + \phi)$$

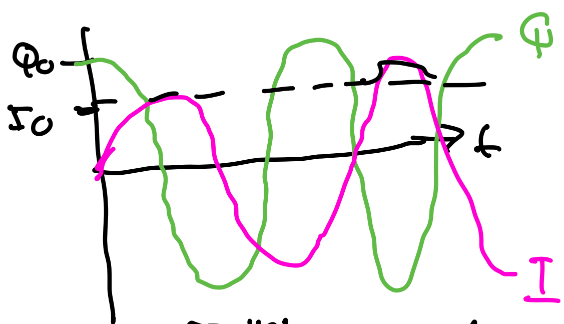
$$I_0 = \omega_0 Q_0$$

From the initial conditions $Q(t=0) = Q_0$ and $I(t=0) = 0$ we can determine ϕ:

$$I(0) = I_0 \sin(\phi) = 0 \Rightarrow \phi = 0$$

Our expressions for Q and I are:

$$Q(t) = Q_0 \cos(\omega_0 t)$$
$$I(t) = I_0 \sin(\omega_0 t)$$



oscillating current
alternating current.

With this we can calculate expressions for electric & magnetic energy:

$$U_E = \frac{Q^2(t)}{2C}$$

$$U_B = \frac{1}{2}LI^2(t)$$

$$U_E = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega t)$$

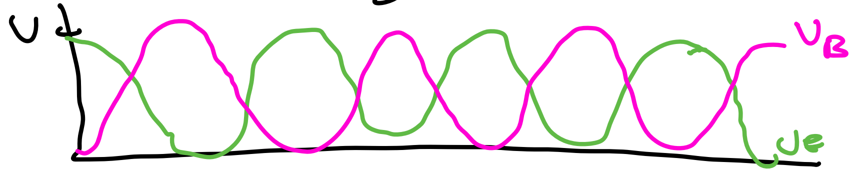
$$U_B = \frac{L I_0^2}{2}\sin^2(\omega t) = \frac{L(-\omega_0 Q_0)^2 \sin^2(\omega_0 t)}{2}$$

$$= \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t)$$

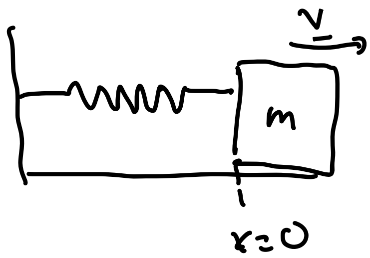
You can verify the total energy remains constant:

$$U = U_E + U_B = \left(\frac{Q_0^2}{2C}\right)\cos^2(\omega_0 t) + \left(\frac{Q_0^2}{2C}\right)\sin^2(\omega_0 t) =$$

$$= \frac{Q_0^2}{2C}$$



this is analogous to a harmonic oscillator.



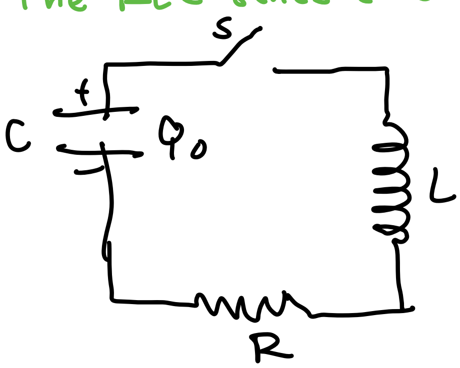
$$U = K + U_p = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Since U is conserved $\frac{dU}{dt} = 0$

$$\frac{dU}{dt} = mv\frac{dv}{dt} + kx\frac{dx}{dt} = 0$$

$$\Rightarrow m\frac{d^2x}{dt^2} + kx = 0$$

The RLC series circuit



Capacitor with charge Q_0

close the switch at $t=0 \rightarrow$ current flows.

Energy will be dissipated by the resistance

$$\frac{dU}{dt} = -I^2R$$

the energy is dissipated $\Rightarrow$ it decreases

Lecture 12

$\frac{\Delta U}{\Delta t} = I\Delta V$ & since $V = RI$

$\Rightarrow \frac{dU}{dt} = RI^2$

And

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt}$$

$$\frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = -I^2R$$

The current is due to a decrease in charge in the capacitor

$$I = -\frac{dQ}{dt}$$

$$\Rightarrow L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \Leftrightarrow L\frac{d^2V}{dt^2} + R\frac{dV}{dt} + \frac{V}{C} = 0$$

(Purcell)

Remember $Q = CV$

In the case of an underdamped oscillation (R small) the solution to this equation is:

$$Q(t) = Q_0 e^{-\gamma t}\cos(\omega' t + \phi)$$

where $\gamma = \frac{R}{2L}$ damping factor and $\omega' = \sqrt{\omega_0^2 - \gamma^2}$ angular freq of oscillations

Q_0 and ϕ are determined by the initial conditions.

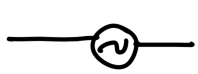
The mechanical analogue is the damped harmonic oscillator

$$m\frac{d^2x}{dt^2} + \underbrace{b\frac{dx}{dt}}_{\text{dissipative force.}} + kx = 0$$

AC circuits

Change in flux of $B \to$ emf

If a coil rotates in a magnetic field, this gives rise to an emf that varies sinusoidally in time $\to$ AC source



$V(t) = V_0 \sin(\omega t)$

$V_0 \equiv$ amplitude (max value).

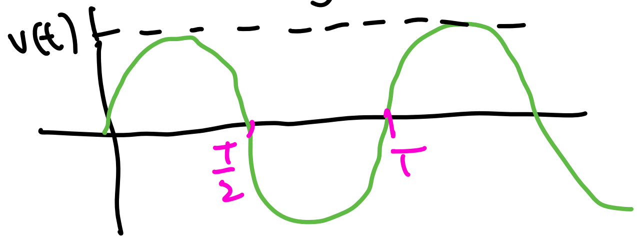
V varies from $-V_0$ to V_0 since $\sin\theta \in [-1, 1]$

In an AC circuit $\to$ current & voltage oscillate without changing their amplitude.

The frequency f of an alternating current is expressed in cycles/second or Hertz (Hz).

cycles → oscillations

The angular frequency $\omega = 2\pi f$ is assumed to be in radians/second.



T is the period

$v(t) = v(t + T)$

and recall $f = \frac{1}{T}$ angular freq $\omega = 2\pi f$

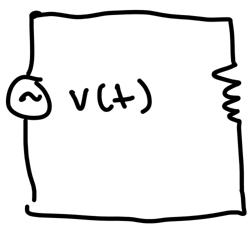
If we connect an alternating voltage source to the RLC circuit, the oscillations will not die out.

An AC will flow into the circuit in response to this driving voltage. The current will be:

$$I(t) = I_0 \sin(\omega t - \phi)$$

The current oscillates with the same frequency as the voltage source and the amplitude of the current is I_0.

Purely resistive load



A resistor connected to an AC generator

$$\boxed{V(t) = V_0 \sin(\omega t)}$$

Apply Kirchhoff's rule:

$$v(t) - V_R(t) = v(t) - \underbrace{I_R(t) R}_{\text{instantaneous voltage drop across the resistor.}} = 0$$

The voltage drop across the resistor

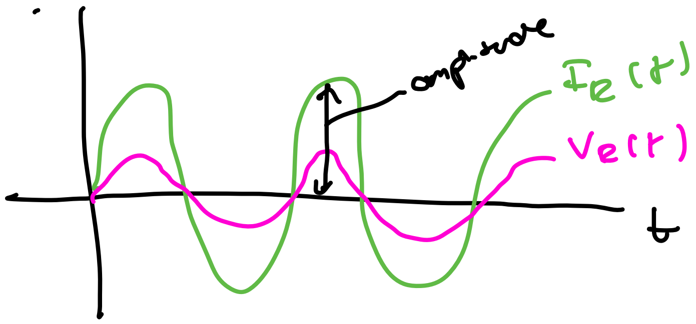
$$V_R(t) = I_R(t) R \Rightarrow I_R(t) = \frac{V_R(t)}{R} = \frac{V_0 \sin(\omega t)}{R}$$

$$= \frac{V_{R_0} \sin(\omega t)}{R} = \boxed{I_{R_0} \sin(\omega t)}$$

We have defined

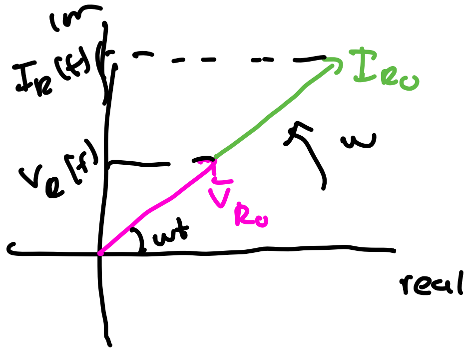
$V_{R_0} = V_0$ and $I_{R_0} = \frac{V_{R_0}}{R}$ is the maximum current.

in this case $\phi = 0 \Rightarrow$ current & voltage are in phase



current & voltage are in phase

We can represent this using a phasor diagram:



phasor is a rotating vector with the following properties

i) length → amplitude

ii) angular speed → vector rotates counterclockwise with an angular speed ω

iii) The projection of the vector along the vertical axis corresponds to the value of the alternating quantity at time t.

The average value of the current over a period.

$$\langle I_R(t)\rangle = \frac{1}{T}\int_0^T I_R(t)\,dt = \frac{1}{T}\int_0^T I_{R0}\sin\omega t\,dt = \frac{I_{R0}}{T}\int_0^T \sin\frac{2\pi t}{T} = 0$$

The average square of the current is non-vanishing:

$$\langle I_R^2(t)\rangle = \frac{1}{T}\int_0^T I_R^2(t)\,dt = I_{R0}^2\frac{1}{T}\int_0^T \sin^2\left(\frac{2\pi t}{T}\right)dt = \frac{1}{2}I_{R0}^2$$

It's convenient to define the root mean square (rms) current:

$$I_{rms} \equiv \left(\langle I_R^2(t)\rangle\right)^{1/2} = \frac{I_{R0}}{\sqrt{2}}$$

Similarly we can define root mean square voltage:

$$V_{rms} \equiv \left(\langle V_R^2(t)\rangle\right)^{1/2}$$

$$= \frac{V_{R0}}{\sqrt{2}}$$

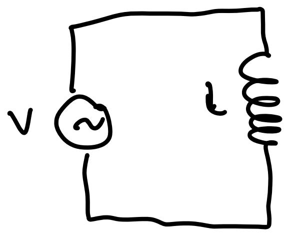
The power dissipated by the resistor is:

$$P_R(t) = I_R(t)V_R(t) = I_R^2(t)R$$ and its average over a period:

$$\langle P_R(t)\rangle = \langle I_R^2(t)R\rangle = \frac{1}{2}I_{R_0}^2 R = I_{rms}^2 R$$

$$= I_{rms}V_{rms} = \frac{V_{rms}^2}{R}$$

Purely inductive load



$$V(t) = V_0 \sin \omega t$$

Applying the modified Kirchhoff's rule for inductors

$$V(t) - V_L(t) = V(t) - L\frac{dI_L}{dt} = 0$$

$$\Rightarrow \frac{dI_L}{dt} = \frac{V(t)}{L} = \frac{V_{L0}\sin\omega t}{L}$$ where $V_{L0} = V_0$

Integrating this we find:

$$I_L(t) = \int dI_L = \frac{V_{L0}}{L}\int \sin\omega t = -\left(\frac{V_{L0}}{\omega t}\right)\cos\omega t$$

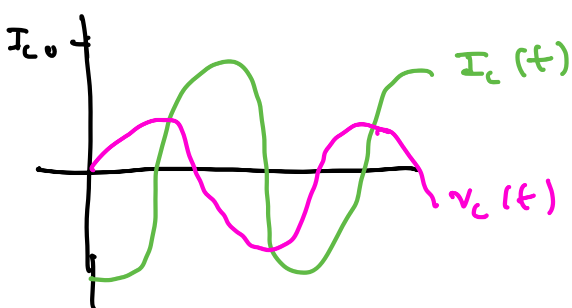
$$= \left(\frac{V_{L0}}{\omega L}\right)\sin\left(\omega t - \frac{\pi}{2}\right)$$ current & voltage are not in phase

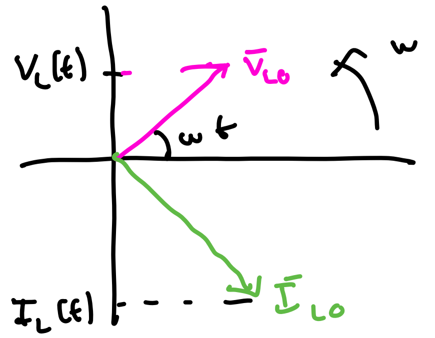
since $-\cos\omega t = \sin\left(\omega t - \frac{\pi}{2}\right)$

$$I_{L0} = \frac{V_{L0}}{\omega L} = \frac{V_{L0}}{X_L}$$ where $X_L \equiv \omega L$ inductive reactance

$[X_L] = \text{Ohms}$ depends linearly with ω

and $\phi = \frac{\pi}{2}$

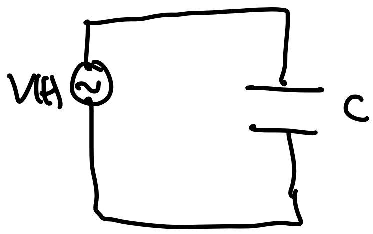




$I_c(t)$ is out of phase with $V_L(t)$ by $\phi = \frac{\pi}{2}$. It reaches its max value after $V_L(t)$ does by $\frac{1}{4}$ cycle

$$\boxed{\text{The current lags the voltage by } \pi/2 \text{ in a purely inductive circuit}}$$

Purely capacitive load



$V(t) = V_0 \sin \omega t$

Using Kirchhoff's rule:

$$V(t) - V_c(t) = V(t) = -\frac{Q(t)}{C} = 0$$

$$\Rightarrow Q(t) = CV(t) = CV_c(t) = CV_{co} \sin \omega t$$

where $V_{co} = V_0$

with this we can get the current

$$I_c(t) = \frac{dQ}{dt} = \omega C V_{co} \cos \omega t \underset{\uparrow}{=} \omega C V_{co} \sin\left(\omega t + \frac{\pi}{2}\right)$$

$\cos \omega t = \sin(\omega t + \frac{\pi}{2})$

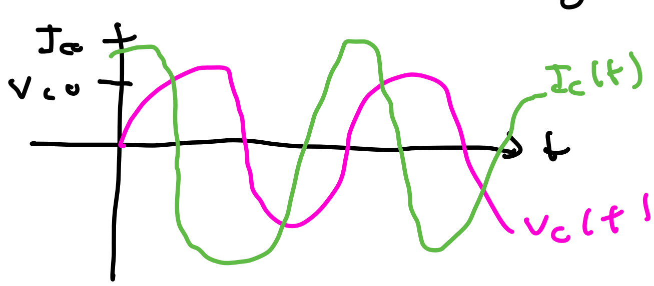
$$I_{co} = \omega C V_{co} = \frac{V_{co}}{X_C}$$

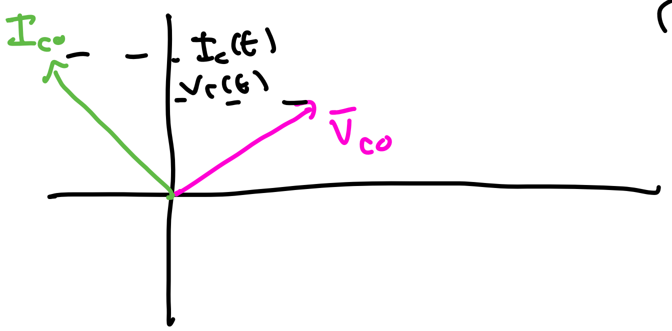
where $X_c \equiv$ capacitance reactance $= \frac{1}{\omega C}$

$\phi = -\frac{\pi}{2}$

$[X_c] = \text{Ohms}$

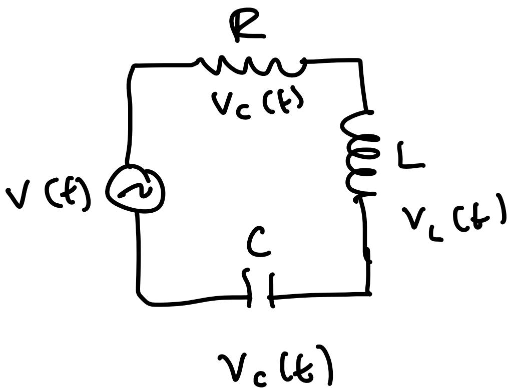
The current & voltage diagrams are:





The current leads the voltage by $\frac{\pi}{2}$ in a capacitive circuit.

The RLC series circuit



Applying the loop rule:

$$V(t) = -V_R(t) - V_L(t) - V_C(t)$$

$$\Rightarrow V(t) - IR - L\frac{dI}{dt} - \frac{Q}{C} = 0$$

$$\Rightarrow L\frac{dI}{dt} + IR + \frac{Q}{C} = V_0 \sin\omega t$$

Assumming initially the capacitor is uncharge $I = \frac{dQ}{dt}$

$$\boxed{L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \sin\omega t}$$

A possible solution $Q(t) = Q_0 \cos(\omega t - \phi)$

where the amplitudes and the phases are:

$$Q_0 = \frac{V_0/L}{\left[(R\omega/L)^2 + (\omega^2 - 1/LC)^2\right]^{1/2}}$$

$$= \frac{V_0}{\omega\sqrt{R^2 + (X_L - X_C)^2}}$$ and

$$\tan\phi = \frac{1}{R}\left(\omega L - \frac{1}{\omega C}\right) = \frac{X_L - X_C}{R}$$

The corresponding current is:

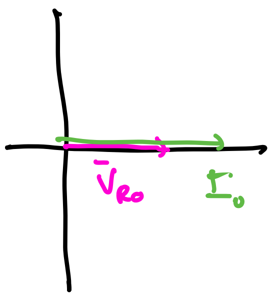
$$I(t) = \frac{dQ}{dt} = I_0 \sin(\omega t - \phi)$$

the amplitude $\quad I_0 = -Q_0\omega = \dfrac{-V_0}{\sqrt{R^2 + (X_L - X_C)^2}}$

* The current has the same amplitude and phase at all points in the RLC circuit
* The instantaneous voltage drop across all elements has a different amplitude & phase relation with the current.

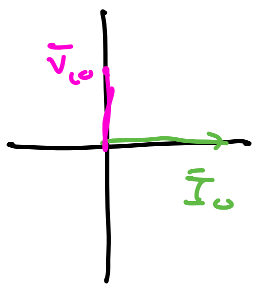
Drawing a phasor diagram:

in the resistor (R)



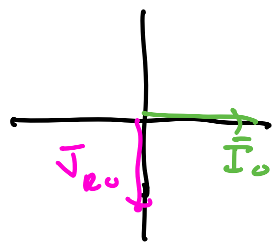
in phase

in the inductor (L)



current lags the voltage by $\pi/2$

in the capacitor (C)



the current leads the voltage by $\pi/2$

We can obtain the instaneous voltages:

$$V_R(t) = I_0 R \sin\omega t = V_{R0}\sin\omega t$$

$$V_L(t) = I_0 X_L \sin\left(\omega t + \frac{\pi}{2}\right) = V_{L0}\cos\omega t$$

$$V_C(t) = I_0 X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -V_{C0}\cos\omega t$$

where

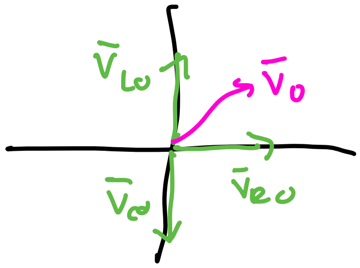
$$V_{R0} = I_0 R \quad , V_{L0} = I_0 X_L \quad , V_{C0} = I_0 X_C$$

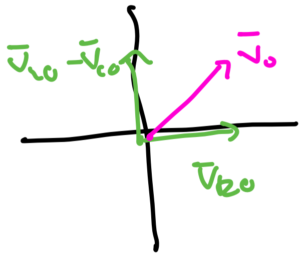
The sum of all 3 is equal to the instaneous voltage supplied by the AC source

$$V(t) = V_R(t) + V_L(t) + V_C(t)$$

Or in phasor representation:

$$\bar{V}_0 = \bar{V}_{R0} + \bar{V}_{L0} + \bar{V}_{C0}$$





$$V_0 = |\bar{V}| = |\bar{V}_{R0} + \bar{V}_{L0} + \bar{V}_{C0}|$$

$$= [V_{R0}^2 + (V_{L0} - V_{C0})^2]^{1/2}$$

$$= [(I_0 R)^2 + (I_0 X_L - I_0 X_C)^2]^{1/2} = I_0 \underbrace{[R^2 + (X_L - X_C)^2]^{1/2}}_{\text{Impedance}}$$

Voltages are not in phase

$V_0 \neq V_{R0} + V_{L0} + V_{C0}$

max amplitude of the source is not equal to the sum of the max amplitudes across each element

$Z = \sqrt{R^2 + (X_L - X_C)^2}$

$[Z] = \text{Ohms}$

effective resistance in an RLC circuit in series.